LIMITS

The title of this book is *From Logic to Limits*, and the first chapter is entitled **Logic**. So, obviously, this last chapter must be called **Limits**.

EXAMPLE 1:

Infinity may not be a number, but it's a pretty interesting concept. Although a variable x cannot actually be infinity, we can nevertheless ask ourselves: What happens to a function as x approaches infinity as a limit?

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Consider the function $y = \frac{1}{x}$.

If $x = 10$, then $y = \frac{1}{10}$.

If $x = 500$, then $y = \frac{1}{500}$.

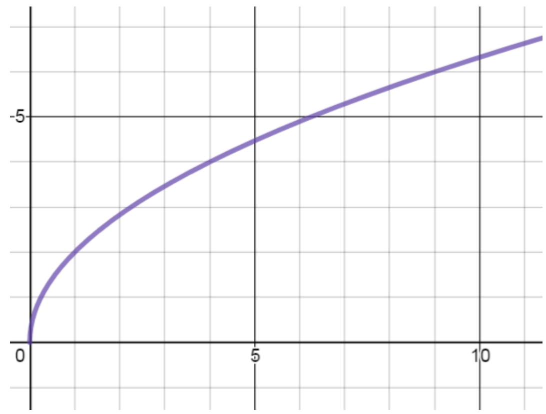
And if we let x be really big, say $x = 1{,}000{,}000$, then $y = \frac{1}{1{,}000{,}000}$.

It seems that as x grows larger and larger, the y-values are getting closer and closer to zero. In fact, we can say that x is approaching infinity, and this is causing y to approach zero. In other words,

As x approaches infinity, y approaches zero:

As $x \to \infty$, $y \to 0$

EXAMPLE 2: **Analyze the following limits for the graph:**



You might recognize this as being the graph of the function $y = \sqrt{x}$.

As $x \to \infty$, $y \to \infty$: Imagine x growing larger and larger. For example, if $x = 100$, then $y = 10$; if $x = 10{,}000$, then $y = 100$. It's clear that, as x grows larger and larger, so does y (though not nearly at the same speed as the x). On the graph, as we move farther and farther to the right, the graph keeps getting higher and higher. In fact, we can rise as high as we desire by choosing x appropriately large. This is why we say: As $x \to \infty$, $y \to \infty$.

As $x \to 0$, $y \to 0$: Note that, since $\sqrt{x}$ is defined only for $x \geq 0$, we'll make sure we use positive values of x that are getting smaller and smaller (approaching 0).

The table indicates that $\sqrt{x}$ approaches **0** also.

x	$\sqrt{x}$
1	1
0.5	0.707
0.25	0.5
0.01	0.1
0.001	0.032
0.0002	0.004

0 0

Limits

<u>Note:</u> It makes no difference at all what happens when $x = 0$. In a limit, we care only about what's happening as x gets closer and closer to the number it's approaching. In fact, we don't care whether or not x is even allowed to be 0.

As $x \to 9$, $y \to 3$: As x gets closer and closer to 9, y gets closer and closer to 3:

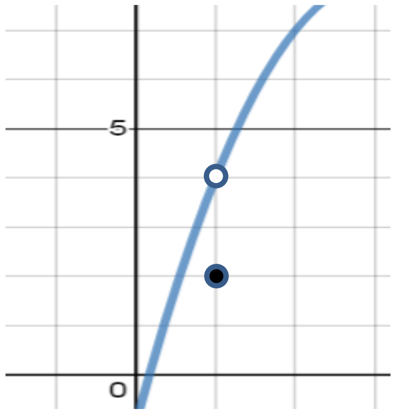
x	8	8.3	8.5	8.8	8.9	8.99	$\longrightarrow$ 9
$\sqrt{x}$	2.828	2.88	2.915	2.966	2.983	2.998	$\longrightarrow$ 3

Final Note This last limit shows that as x approaches 9, y approaches 3. It is also the case that when $x = 9$, $y = 3$. Yet, this fact has NOTHING to do with the limit. Even if y were a hundred trillion when $x = 3$, the limit statement would still be correct.

> "The educated differ from
> the uneducated
> as much as the living from
> the dead."
>
> □□□□□□□□□□(Aristotle)

Bottom line: Functional values are one thing, **a**nd limits are another. The following example might clarify this critical distinction. It also gives us limits which do <u>not</u> involve infinity.

EXAMPLE 3: **Let f be defined by the graph**



a. $f(1) =$ _____

b. As $x \to 1$, $y \to$ _____

Solution:

a. Looking at the solid dot part of the graph, we conclude that $f(1) = \mathbf{2}$.

b. As x approaches 1, the graph is heading toward a y-value of 4. The open circle at the point (1, 4) means that x never reaches 1 on the curve. Still, as x approaches 1, y approaches a value of **4**. That is, as $\boldsymbol{x \to 1, y \to 4}$

To summarize, at $x = 1$, the function value is 2. But as $x \to 1$, $y \to 4$. So we note again that calculating limits has nothing at all to do with what might happen if x actually becomes the value it's approaching. In fact, in the graph of this example, if we were to remove the solid dot, the answer to part a. would change to "Undefined," but the limit would remain exactly as it is.

❑ DEFINITION AND NOTATION

You should now be familiar with the following notation:
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$\lim\limits_{x \to a^+} f(x) = L$	As x approaches a from the right, $f(x)$ is approaching L.
$\lim\limits_{x \to a^-} f(x) = L$	As x approaches a from the left, $f(x)$ is approaching L.

Now for a new idea: a two-sided limit. We write

$$\lim_{x \to a} f(x) = L$$

when we want to indicate the following:

As x approaches a from either direction,

$f(x)$ is approaching L.

We can thus write:

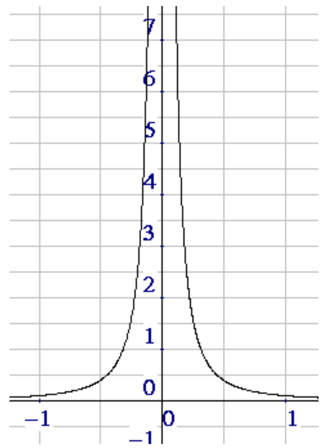
IF $\lim\limits_{x \to a^+} f(x) = L$ AND $\lim\limits_{x \to a^-} f(x) = L$

THEN $\lim\limits_{x \to a} f(x) = L$.

In words, we say that if both of the one-sided limits exist and are equal, then the two-sided limit exists and its value is the same as the two one-sided limits.

❑ LIMITS FROM GRAPHS

EXAMPLE 1: **Calculate all the limits as $x \to 0$:**



Solution: We see from the right side of the graph that as $x \to 0^+$, $f(x) \to \infty$. That is,

$$\lim_{x \to 0^+} f(x) = \infty$$

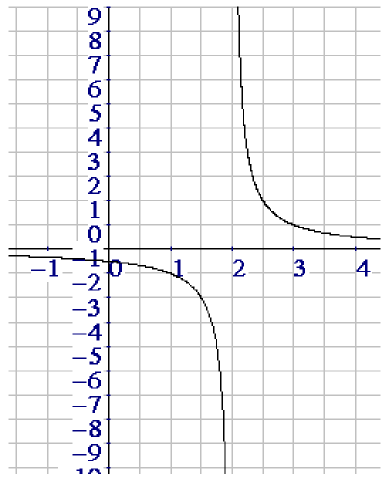
Also, as $x \to 0^-$, $f(x) \to \infty$, so we also have the fact that

$$\lim_{x \to 0^-} f(x) = \infty$$

By the definition of the two-sided limit, we conclude that

$$\boxed{\lim_{x \to 0} f(x) = \infty}$$

EXAMPLE 2: **Calculate all the limits as $x \to 2$:**



Solution: The part of the curve in Quadrant I tells us that

$$\lim_{x \to 2^+} f(x) = \infty,$$

whereas the part of the curve in Quadrant IV indicates that

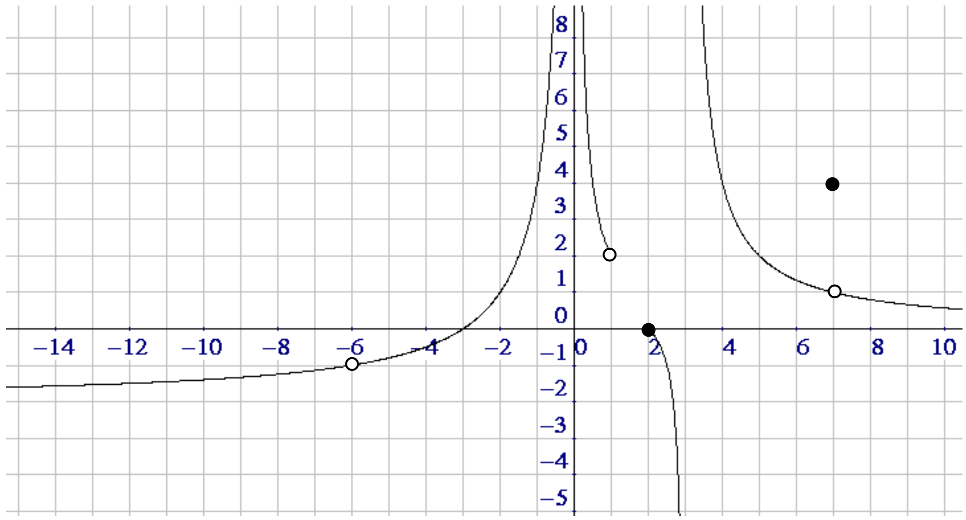
$$\lim_{x \to 2^-} f(x) = -\infty.$$

Thus, each one-sided limit exists, but since they are not equal, the two-sided limit

$$\boxed{\lim_{x \to 2} f(x) \text{ does not exist}}$$

Homework

Let *f* be defined by the following graph:



1. a. Explain why *f* is a function.

   b. Find the doman of *f*.

   c. Find the range of *f*.

2. a. $\lim_{x\to\infty} f(x)$ b. $\lim_{x\to-\infty} f(x)$

3. a. $f(-6)$ b. $\lim_{x\to-6^-} f(x)$ c. $\lim_{x\to-6^+} f(x)$ d. $\lim_{x\to-6} f(x)$

4. a. $f(0)$ b. $\lim_{x\to0^-} f(x)$ c. $\lim_{x\to0^+} f(x)$ d. $\lim_{x\to0} f(x)$

5. a. $f(1)$ b. $\lim_{x\to1^-} f(x)$ c. $\lim_{x\to1^+} f(x)$ d. $\lim_{x\to1} f(x)$

6. a. $f(2)$ b. $\lim_{x\to2^-} f(x)$ c. $\lim_{x\to2^+} f(x)$ d. $\lim_{x\to2} f(x)$

7. a. $f(3)$ b. $\lim\limits_{x \to 3^-} f(x)$ c. $\lim\limits_{x \to 3^+} f(x)$ d. $\lim\limits_{x \to 3} f(x)$

8. a. $f(7)$ b. $\lim\limits_{x \to 7^-} f(x)$ c. $\lim\limits_{x \to 7^+} f(x)$ d. $\lim\limits_{x \to 7} f(x)$

❑ *LIMITS FROM FORMULAS*

<u>EXAMPLE 3:</u> **Find the value of each of the following limits:**

A. $\lim\limits_{x \to 7} 12 = \mathbf{12}$, since 12 is certainly infinitely close to 12 as x approaches 7 (in fact, it doesn't matter what x approaches).

B. $\lim\limits_{x \to 2} 3x = \mathbf{6}$, because as x gets closer and closer to 2 (from either side), $3x$ gets closer and closer to 6.

C. $\lim\limits_{x \to 0} 7x + 9 = \mathbf{9}$

D. $\lim\limits_{x \to 3} x^2 - x + 5 = \mathbf{11}$

E. $\lim\limits_{x \to 3} \frac{1}{x^2} = \mathbf{\frac{1}{9}}$

F. $\lim\limits_{x \to 16} \sqrt{x} = \mathbf{4}$

G. $\lim\limits_{x \to 0} \sqrt{x}$ *does not exist*

Here's why: Recall that "$x \to 0$" means that x approaches 0 from <u>both</u> sides — and if both the left and right-hand limits exist and are equal, then we can say that the limit exists. But in this case, the left-hand limit does not exist (it's outside the domain of the square root function), and therefore the two-sided limit does not exist.

H. $\lim\limits_{x \to 0} 2x^3 - x^2 + 8x - 9 \;=\; \mathbf{-9}$

<u>EXAMPLE 4:</u> **Find the value of each of the following limits:**

A. $\lim\limits_{x \to 0} \dfrac{x^2 + x}{x}$

As x shrinks toward 0, the top approaches 0, and so does the bottom, leaving us with the fraction $\frac{0}{0}$, which is meaningless. But watch what some algebra can do:

$$\frac{x^2 + x}{x} \;=\; \frac{x(x+1)}{x} \;=\; \frac{\cancel{x}(x+1)}{\cancel{x}} \;=\; x + 1$$

Note: We can cancel out the x if we know for sure that it can never be zero. But this is clearly the case, since x is approaching zero, and therefore can <u>never</u> be zero.

So we can change the original problem to $\lim\limits_{x \to 0} x + 1$, which is simply 1. Thus,

$$\boxed{\lim_{x \to 0} \frac{x^2 + x}{x} \;=\; 1}$$

B. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

As x approaches 3, both the top and bottom of the fraction approach 0, resulting in the same meaningless fraction as the preceding example. But,

$$\frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{x-3} = \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}} = x+3,$$

Therefore, the original problem is equal to $\lim\limits_{x\to 3} x+3$, which has a value of 6. Thus,

$$\boxed{\lim_{x\to 3}\frac{x^2-9}{x-3} = 6}$$

C. $\lim\limits_{x\to -4}\dfrac{x+4}{x^2-16}$

The top and bottom of the fraction are each approaching 0 as x approaches –4. However,

$$\frac{x+4}{x^2-16} = \frac{x+4}{(x+4)(x-4)} = \frac{\cancel{x+4}}{\cancel{(x+4)}(x-4)} = \frac{1}{x-4}, \text{ so}$$

$$\lim_{x\to -4}\frac{x+4}{x^2-16} = \lim_{x\to -4}\frac{1}{x-4} = \boxed{-\frac{1}{8}}$$

D. $\lim\limits_{x\to 25}\dfrac{x-25}{\sqrt{x}-5}$

The top and bottom are each approaching 0 — this is undesirable — and in this problem, we have two ways to remedy the situation:

The 1st method for changing the fraction involves factoring:

$$\frac{x-25}{\sqrt{x}-5} = \frac{(\sqrt{x}+5)(\sqrt{x}-5)}{\sqrt{x}-5} = \frac{(\sqrt{x}+5)\cancel{(\sqrt{x}-5)}}{\cancel{\sqrt{x}-5}} = \sqrt{x}+5$$

The 2nd method for changing the fraction involves rationalizing the denominator:

$$\frac{x-25}{\sqrt{x}-5} = \frac{x-25}{\sqrt{x}-5}\left[\frac{\sqrt{x}+5}{\sqrt{x}+5}\right] = \frac{(x-25)(\sqrt{x}+5)}{x-25} = \sqrt{x}+5$$

Either way, $\lim\limits_{x \to 25} \dfrac{x-25}{\sqrt{x}-5} \;=\; \lim\limits_{x \to 25} \sqrt{x}+5 \;=\; 5+5 \;=\; \boxed{10}$

E. $\lim\limits_{x \to 7} \dfrac{x^2-49}{x+5}$

As x approaches 7, the top approaches 0, but the bottom approaches 12, giving us the fraction $\frac{0}{12}$, which is 0. There's no real problem here — no fancy algebra to perform. Thus,

$$\lim_{x \to 7} \frac{x^2-49}{x+5} \;=\; \boxed{0}$$

Homework

9. $\lim\limits_{x \to 99} 7$

10. $\lim\limits_{x \to 3} 2x+10$

11. $\lim\limits_{x \to -1} -x^2-x+7$

12. $\lim\limits_{x \to 7} \dfrac{2}{x}$

13. $\lim\limits_{x \to 0^+} \sqrt{x}$

14. $\lim\limits_{x \to 0} \sqrt{x}$

15. $\lim\limits_{x \to 24} \sqrt{x+1}$

16. $\lim\limits_{x \to 0} \dfrac{x^2-x}{x}$

17. $\lim\limits_{x \to 5} \dfrac{x^2-25}{x-5}$

18. $\lim\limits_{x \to -3} \dfrac{x+3}{x^2-9}$

19. $\lim\limits_{x \to 4} \dfrac{x+4}{x^2+16}$

20. $\lim\limits_{x \to 100} \dfrac{x-100}{\sqrt{x}-10}$

Solutions

1. a. Given any x in the domain of the function, there exists <u>exactly one</u> y-value for it. In other words, it satisfies the Vertical Line Test: no vertical line cuts the graph in more than one place.

b. Domain = $(-\infty, -6) \sim (-6, 0) \sim (0, 1) \sim [2, 3) \sim (3, \infty)$

c. Range = ⊕, or in interval notation, $(-\infty, \infty)$.

2. a. 0 b. –2

3. a. Does not exist (DNE) b. –1 c. –1 d. –1

4. a. DNE b. ∞ c. ∞ d. ∞

5. a. DNE b. 2 c. DNE d. DNE

6. a. 0 b. DNE c. 0 d. DNE

7. a. DNE b. $-\infty$ c. ∞ d. DNE

8. a. 4 b. 1 c. 1 d. 1

9. 7 **10.** 16 **11.** 7 **12.** $\frac{2}{7}$ **13.** 0 **14.** DNE

15. 5 **16.** –1 **17.** 10 **18.** $-\frac{1}{6}$ **19.** $\frac{1}{4}$ **20.** 20